On parking functions and their combinatorial statistics

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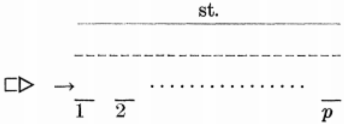
Preliminaries on parking functions

Introduced in Kolheim & Weiss, An occupancy discipline and applications, SIAM J. Appl. Math. 14 (1966), 1266-1274.

6. A parking problem—the case of the capricious wives. Let st. be a street with p parking places. A car



occupied by a man and his dozing wife enters st. at the left and moves towards the right. The wife awakens at a capricious moment and orders her husband to park immediately! He dutifully parks at his present location, if it is empty, and if not, continues to the right and parks at the next available space. If no space is available he leaves st.

Preliminaries on parking functions

Examples of parking functions:

- Any permutation $\pi \in \mathfrak{S}_n$ is a parking function sending the i-th car to the π_i-th parking spot.
- The parking preference $(1, 1, \ldots, 1)$ is a parking function sending the i-th car to the i-th parking spot.

Examples of not parking functions:

- The preference $\underbrace{(n, n, \ldots, n)}_{n \text{ times}}$ fails to be a parking function.
- The preference $(2, 3, 2)$ fails to be a parking function.

Descriptions of parking functions

Definition: Let a_i be a parking preference, b_i its weakly-increasing rearrangement. Then a_i is a parking function iff $b_i \leq i$.

Equivalently: We call a_i a parking function iff $\text{card}\{a_i \leq j\} \geq j$.

Remark: Permutations of parking functions are parking functions.

Number of parking functions

The number of parking functions of length n is $(n+1)^{n-1}$. [2]

Pollak's Proof:

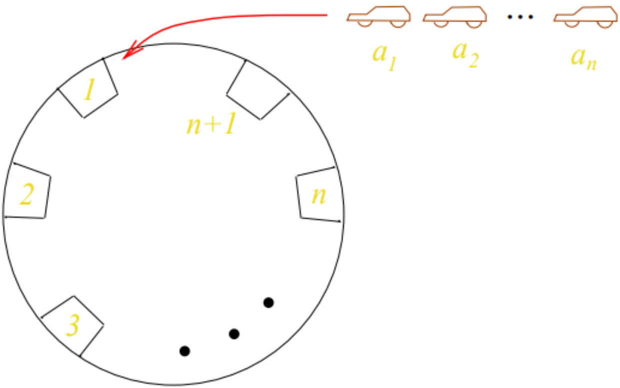


Image from [2]

Number of parking functions

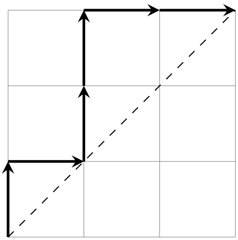
The number $(n+1)^{n-1}$ shows up in all different contexts. Parking functions of length n are equinumerous with:

- Trees on $n+1$ labelled vertices [11]
- Rooted forests on n labelled vertices
  - Bijection given by Schützenberger 1968
- Monomials in the Hilbert series of diagonal harmonics [4]
- Labelled Dyck paths of length $2n$ [4]

Preliminaries on Dyck paths

Definition: A Dyck path is a lattice path from $(0, 0)$ to (n, n) of north and east steps above the line $y = x$.

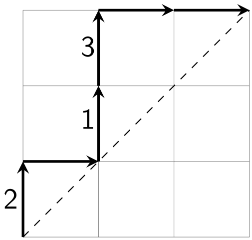
Example:



Preliminaries on Dyck paths

Definition: A labelled Dyck path is a Dyck path with north steps labelled 1 through n with ascending columns.

Example:

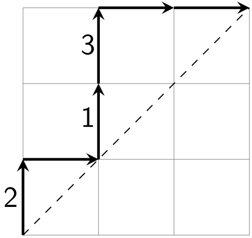


Preliminaries on Dyck paths

Unlabelled: The number of unlabelled Dyck paths of length $2n$ is the n-th Catalan number. [10]

Labelled: The number of labelled Dyck paths of length $2n$ is the number of parking functions of length n. [4]

Example: Consider the parking function $(2, 1, 2)$.



Combinatorial statistics

Definition: A statistic on a set S is a function $S \to \mathbb{N}$ assigning to each element s of S a natural number n in $\mathbb{N}$.

Example 1: The inversion statistic inv: $\mathfrak{S}_n \to \mathbb{N}$ counts the number of inversions of a permutation. So $\text{inv}(312) = 2$.

Example 2: The major statistic maj: $\mathfrak{S}_n \to \mathbb{N}$ sums the positions of descents of a permutation. So $\text{maj}(321) = 1 + 2 = 3$.

Combinatorial statistics

Definition: The q-analogue of n is the polynomial

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}$$

which converges to n in the limit $q \to 1$. [3]

Theorem:

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} = [n]_q!$$

where $[n]_q!$ is the q-factorial $[n]_q[n-1]_q \cdots [1]_q$. [3]

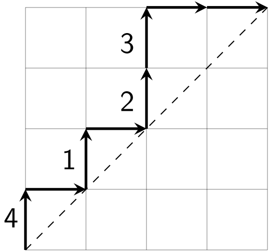
Statistics on parking functions

<u>Definition:</u> The area statistic counts the full boxes between the labelled Dyck path and the line $y = x$.

<u>Definition:</u> The dinv statistic counts ascents in the main diagonal and the descents from one diagonal to the next.

<u>Example:</u>

area 1



dinv 2

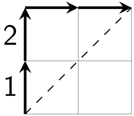
Statistics on parking functions

Theorem:

$$\sum_{\pi \in PF_n} q^{\mathrm{dinv}(\pi)} = \sum_{\pi \in PF_n} q^{\mathrm{area}(\pi)} \qquad [4]$$

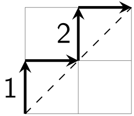
Example: Consider parking functions of length 2.

area 1



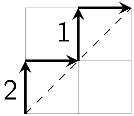
dinv 0

area 0



dinv 1

area 0



dinv 0

Statistics on parking functions

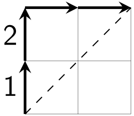
Theorem:

$$\sum_{\pi \in PF_n} q^{\operatorname{dinv}(\pi)} = \sum_{\pi \in PF_n} q^{\operatorname{area}(\pi)} \qquad [4]$$

$$= 2 + q$$

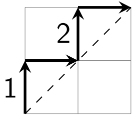
Example: Consider parking functions of length 2.

area 1



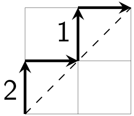
dinv 0

area 0



dinv 1

area 0



dinv 0

Diagonal harmonics

For any $\sigma \in \mathfrak{S}_n$ define $R_s > \cdots > R_2 > R_1 > 0$ to be the ascending runs and $w_k(\sigma)$ the number of symbols greater than σ_k in its run plus the number of symbols less than σ_k in the next run.

For $\sigma = (456231)$, $456 > 23 > 1 > 0$ and $w_5(\sigma) = 1 + 2 = 3$.

Consider objects $(\sigma; u_1, u_2, \ldots, u_n)$ with $\sigma \in S_n$ and $u_k < w_k(\sigma)$.

Notice that

$$\sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{maj}(\sigma)} \prod_{k=1}^{n} [w_k(\sigma)]_q$$

is a generating function for such objects. [4]

Diagonal harmonics

Fermionic formula for the Hilbert series of diagonal harmonics:

$$CH_n(q,t) = \sum_{\pi \in PF_n} q^{\mathrm{dinv}(\pi)} t^{\mathrm{area}(\pi)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{maj}(\sigma)} \prod_{k=1}^{n} [w_k(\sigma)]_q$$

$$= \sum_{I \in \mathcal{I}_n} t^{\mathrm{tstat}(I)} q^{\mathrm{qstat}(I)} = \sum_{\pi \in PF_n} q^{\mathrm{area}(\pi)} t^{\mathrm{pmaj}(\pi)} \qquad [4]$$

Specializing to $CH_n(q,1)$ gives the theorem from before,

$$CH_n(q,1) = \sum_{\pi \in PF_n} q^{\mathrm{dinv}(\pi)} = \sum_{\pi \in PF_n} q^{\mathrm{area}(\pi)} \qquad [4]$$

Generalizations of parking functions

- Naples parking functions allow cars to try to park one spot back before moving on down the street. [5]
- k-Naples parking functions generalize these by allowing cars to park k spots back before moving on. [6]
- Parking sequences generalize let the size of cars vary. [7]
- Trailer sequences have a trailer blocking the first spots. [8]

Generalized parking functions are a pain

- The number of parking sequences with car sizes y_i is

$$(y_1 + n) \cdot (y_1 + y_2 + n - 1) \cdots (y_1 + \cdots + y_{n-1} + 2)$$

which simplifies to $(n+1)^{n-1}$ when $y_i = 1$. [7]

- The number $|PF_{n,k}|$ of k-Naples parking functions satisfies

$$|PF_{n+1,k}| = \sum_{i=0}^{n} \binom{n}{i} \min((i+1)+k, n+1)|PF_{i,k}|(n-i+1)^{n-i-1}$$

which has no known closed formula. [6]

Potential research directions

- Generalize or invent new statistics for these generalized parking functions and enumerate them.
- Conjucture and find bijective proofs of identities involving these generalizations or inventions.
- Use these statistical methods to answer open problems.
  - For instance, consider a decreasing k-Naples parking function. Which rearrangements are also k-Naples? [6]
  - Conjecture: All rearrangements of a parking preference is Naples if it only crosses $y = x$ at one corner. [6]

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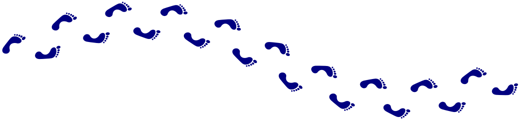

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References and further reading

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